

Variable Order Mittag–Leffler Fractional Operators on Isolated Time Scales and Application to the Calculus of Variations



Thabet Abdeljawad, Raziye Mert and Delfim F. M. Torres 

Abstract We introduce new fractional operators of variable order in isolated time scales with Mittag–Leffler kernels. This allows a general formulation of a class of fractional variational problems involving variable-order difference operators. Main results give fractional integration by parts formulas and necessary optimality conditions of Euler–Lagrange type.

Keywords Fractional calculus · Atangana–Baleanu fractional derivative · Fractional variational problems

1 Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to an arbitrary non-integer order. It has been used effectively in the modeling of many problems in various fields of science and engineering, reflecting successfully the description of non-local properties of complex systems [12, 42]. For the sake of finding more fractional operators with different kernels, recently several authors have introduced and studied new non-local derivatives with non-singular kernels and have applied them successfully to some real world problems [2, 3, 8–11, 19, 20, 27–29, 33, 34]. What makes those fractional derivatives with Mittag–Leffler kernels

T. Abdeljawad (✉)

Department of Mathematics and General Sciences, Prince Sultan University,
P.O. Box 66833, Riyadh 11586, Saudi Arabia
e-mail: tabdeljawad@psu.edu.sa

R. Mert

Mechatronic Engineering Department, University of Turkish Aeronautical Association,
06790 Ankara, Turkey
e-mail: rmert@thk.edu.tr

D. F. M. Torres

Department of Mathematics, Center for Research and Development in Mathematics
and Applications (CIDMA), University of Aveiro, 3810-193 Aveiro, Portugal
e-mail: delfim@ua.pt

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J. F. Gómez et al. (eds.), *Fractional Derivatives with Mittag–Leffler Kernel*,
Studies in Systems, Decision and Control 194,
https://doi.org/10.1007/978-3-030-11662-0_3

more interesting is that their corresponding fractional integrals contain Riemann–Liouville fractional integrals as part of their structure. Moreover, such operators enable numerical analysts to develop more efficient algorithms in solving fractional dynamical systems by concentrating only on the coefficients of the differential equations rather than worrying about the singularity of the kernels, as in the case of classical fractional operators [7].

In 1993, Samko and Ross investigated integrals and derivatives not of a constant but of variable order [39–41]. Afterwards, several pure mathematical and applicational papers contributed to the theory of variable order fractional calculus [6, 21–26, 32, 37, 38, 44, 50]. Here we continue this line of research.

The article is organized as follows. In Sect. 2, we introduce new definitions of two different types of left and right nabla fractional sums of variable order, two different types of discrete versions of the left and right generalized fractional integral operators, together with two different types of fractional sums and differences of variable order in the sense of Atangana–Baleanu. Afterwards, in Sect. 3, we prove integration by parts formulas for Atangana–Baleanu fractional sums and differences with variable order. We end with Sect. 4, applying our results to the calculus of variations.

2 Fractional Sums and Differences of Variable Order

The study of fractional calculus on time scales was initiated with the papers [13–15] and is now under strong development: see, e.g., [16–18, 36, 43]. Here, inspired by the results of [3, 8], we introduce new nabla fractional operators of variable order in isolated time scales. The reader interested on the motivation and importance to consider variable order operators is referred to [46–48] and references therein.

Let $a, b \in \mathbb{R}$ with $b - a$ a positive integer. The sets $\mathbb{N}_a$, ${}_b\mathbb{N}$, and $\mathbb{N}_{a,b}$ are defined by

$$\mathbb{N}_a = \{a, a + 1, a + 2, \dots\}, \quad {}_b\mathbb{N} = \{\dots, b - 2, b - 1, b\}, \quad \mathbb{N}_{a,b} = \{a, a + 1, a + 2, \dots, b\},$$

respectively. Our operators use the concepts of *rising function* and *discrete Mittag–Leffler function*.

Definition 1 (*Rising function* [30]) (i) For a natural number m and $t \in \mathbb{R}$, the m rising (ascending) factorial of t is defined by

$$t^{\overline{m}} = \prod_{k=0}^{m-1} (t + k), \quad t^{\overline{0}} = 1.$$

(ii) For any real number α , the (generalized) rising function is defined by

$$t^{\overline{\alpha}} = \frac{\Gamma(t + \alpha)}{\Gamma(t)}, \quad t \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}, \quad 0^{\overline{\alpha}} = 0.$$

Definition 2 (*Nabla discrete Mittag–Leffler function* [1, 4]) For $\lambda \in \mathbb{R}$, $|\lambda| < 1$ and $\alpha, \beta, z \in \mathbb{C}$ with $\operatorname{Re}(\alpha) > 0$, the nabla discrete Mittag–Leffler function is defined by

$$E_{\alpha, \beta}(\lambda, z) = \sum_{k=0}^{\infty} \lambda^k \frac{z^{\overline{k\alpha + \beta - 1}}}{\Gamma(\alpha k + \beta)}.$$

For $\beta = 1$, we write

$$E_{\overline{\alpha}}(\lambda, z) \triangleq E_{\overline{\alpha}, 1}(\lambda, z) = \sum_{k=0}^{\infty} \lambda^k \frac{z^{\overline{k\alpha}}}{\Gamma(\alpha k + 1)}.$$

To start, we define two different types of nabla fractional sums of variable order.

Definition 3 (*Left nabla fractional sums of order $\alpha(t)$ — types I and II*) Let $0 < \alpha(t) \leq 1$ for all $t \in \mathbb{N}_a$. For a function $f : \mathbb{N}_a \rightarrow \mathbb{R}$,

1. the type I left nabla fractional sum of order $\alpha(t)$ is defined by

$${}_a \nabla^{-\alpha(t)} f(t) = \frac{1}{\Gamma(\alpha(t))} \sum_{s=a+1}^t (t - \rho(s))^{\overline{\alpha(t)-1}} f(s), \quad t \in \mathbb{N}_{a+1};$$

2. the type II left nabla fractional sum of order $\alpha(t)$ is defined by

$${}_a^* \nabla^{-\alpha(t)} f(t) = \sum_{s=a+1}^t (t - \rho(s))^{\overline{\alpha(s)-1}} f(s) \frac{1}{\Gamma(\alpha(s))}, \quad t \in \mathbb{N}_{a+1}.$$

Definition 4 (*Right nabla fractional sums of order $\alpha(t)$ — types I and II*) Let $0 < \alpha(t) \leq 1$ for all $t \in {}_b\mathbb{N}$. For a function $f : {}_b\mathbb{N} \rightarrow \mathbb{R}$,

1. the type I right nabla fractional sum of order $\alpha(t)$ is defined by

$$\nabla_b^{-\alpha(t)} f(t) = \frac{1}{\Gamma(\alpha(t))} \sum_{s=t}^{b-1} (s - \rho(t))^{\overline{\alpha(t)-1}} f(s), \quad t \in {}_{b-1}\mathbb{N};$$

2. the type II right nabla fractional sum of order $\alpha(t)$ is defined by

$${}_b^* \nabla^{-\alpha(t)} f(t) = \sum_{s=t}^{b-1} (s - \rho(t))^{\overline{\alpha(s)-1}} f(s) \frac{1}{\Gamma(\alpha(s))}, \quad t \in {}_{b-1}\mathbb{N}.$$

Following [3], we now define two different discrete versions of the left and right generalized fractional integral operators.

Definition 5 (*Discrete left generalized fractional integral operators — types I and II*) Let $0 < \alpha(t) < 1/2$ for all $t \in \mathbb{N}_a$. For a function $\varphi : \mathbb{N}_a \rightarrow \mathbb{R}$,

1. the type I discrete left generalized fractional integral operator is defined by

$$\mathbb{E}_{\alpha(t), 1, \frac{-\alpha(t)}{1-\alpha(t)}, a^+} \varphi(t) = \frac{B(\alpha(t))}{1-\alpha(t)} \sum_{s=a+1}^t E_{\alpha(t)} \left[\frac{-\alpha(t)}{1-\alpha(t)}, t - \rho(s) \right] \varphi(s), \quad t \in \mathbb{N}_{a+1}; \quad (1)$$

2. the type II discrete left generalized fractional integral operator is defined by

$$\mathcal{E}_{\alpha(t), 1, \frac{-\alpha(t)}{1-\alpha(t)}, a^+} \varphi(t) = \sum_{s=a+1}^t \frac{B(\alpha(s))}{1-\alpha(s)} E_{\alpha(s)} \left[\frac{-\alpha(s)}{1-\alpha(s)}, t - \rho(s) \right] \varphi(s), \quad t \in \mathbb{N}_{a+1}. \quad (2)$$

Definition 6 (*Discrete right generalized fractional integral operators — types I and II*) Let $0 < \alpha(t) < 1/2$ for all $t \in {}_b\mathbb{N}$. For a function $\varphi : {}_b\mathbb{N} \rightarrow \mathbb{R}$,

1. the type I discrete right generalized fractional integral operator is defined by

$$\mathbb{E}_{\alpha(t), 1, \frac{-\alpha(t)}{1-\alpha(t)}, b^-} \varphi(t) = \frac{B(\alpha(t))}{1-\alpha(t)} \sum_{s=t}^{b-1} E_{\alpha(t)} \left[\frac{-\alpha(t)}{1-\alpha(t)}, s - \rho(t) \right] \varphi(s), \quad t \in {}_{b-1}\mathbb{N}; \quad (3)$$

2. the type II discrete right generalized fractional integral operator is defined by

$$\mathcal{E}_{\alpha(t), 1, \frac{-\alpha(t)}{1-\alpha(t)}, b^-} \varphi(t) = \sum_{s=t}^{b-1} \frac{B(\alpha(s))}{1-\alpha(s)} E_{\alpha(s)} \left[\frac{-\alpha(s)}{1-\alpha(s)}, s - \rho(t) \right] \varphi(s), \quad t \in {}_{b-1}\mathbb{N}. \quad (4)$$

We now define two different types of fractional sums and differences of variable order in the sense of Atangana–Baleanu [8] (the so-called AB operators).

Definition 7 (*Left AB nabla fractional sums of order $\alpha(t)$ — types I and II*) Let $0 < \alpha(t) \leq 1$ for all $t \in \mathbb{N}_a$. For a function $f : \mathbb{N}_a \rightarrow \mathbb{R}$,

1. the type I left AB nabla fractional sum of order $\alpha(t)$ is defined by

$$\begin{aligned} {}^aAB \nabla^{-\alpha(t)} f(t) &= \frac{1-\alpha(t)}{B(\alpha(t))} f(t) + \frac{\alpha(t)}{B(\alpha(t))\Gamma(\alpha(t))} \sum_{s=a+1}^t (t - \rho(s))^{\overline{\alpha(t)-1}} f(s) \\ &= \frac{1-\alpha(t)}{B(\alpha(t))} f(t) + \frac{\alpha(t)}{B(\alpha(t))} {}_a \nabla^{-\alpha(t)} f(t), \quad t \in \mathbb{N}_{a+1}; \end{aligned} \quad (5)$$

2. the type II left AB nabla fractional sum of order $\alpha(t)$ is defined by

$$\begin{aligned} {}^*{}_aAB \nabla^{-\alpha(t)} f(t) &= \frac{1-\alpha(t)}{B(\alpha(t))} f(t) + \sum_{s=a+1}^t \frac{\alpha(s)}{B(\alpha(s))\Gamma(\alpha(s))} (t - \rho(s))^{\overline{\alpha(s)-1}} f(s) \\ &= \frac{1-\alpha(t)}{B(\alpha(t))} f(t) + {}^*_a \nabla^{-\alpha(t)} \frac{\alpha f}{B \circ \alpha}(t), \quad t \in \mathbb{N}_{a+1}. \end{aligned} \quad (6)$$

Definition 8 (*Right AB nabla fractional sums of order $\alpha(t)$ — types I and II*) Let $0 < \alpha(t) \leq 1$ for all $t \in {}_b\mathbb{N}$. For a function $f : {}_b\mathbb{N} \rightarrow \mathbb{R}$,

1. the type *I* right *AB* nabla fractional sum of order $\alpha(t)$ is defined by

$$\begin{aligned} {}^{AB}\nabla_b^{-\alpha(t)} f(t) &= \frac{1 - \alpha(t)}{B(\alpha(t))} f(t) + \frac{\alpha(t)}{B(\alpha(t))\Gamma(\alpha(t))} \sum_{s=t}^{b-1} (s - \rho(t))^{\overline{\alpha(t)-1}} f(s) \\ &= \frac{1 - \alpha(t)}{B(\alpha(t))} f(t) + \frac{\alpha(t)}{B(\alpha(t))} \nabla_b^{-\alpha(t)} f(t), \quad t \in {}_{b-1}\mathbb{N}; \end{aligned} \quad (7)$$

2. the type *II* right *AB* nabla fractional sum of order $\alpha(t)$ is defined by

$$\begin{aligned} {}^{*AB}\nabla_b^{-\alpha(t)} f(t) &= \frac{1 - \alpha(t)}{B(\alpha(t))} f(t) + \sum_{s=t}^{b-1} \frac{\alpha(s)}{B(\alpha(s))\Gamma(\alpha(s))} (s - \rho(t))^{\overline{\alpha(s)-1}} f(s) \\ &= \frac{1 - \alpha(t)}{B(\alpha(t))} f(t) + {}^*\nabla_b^{-\alpha(t)} \frac{\alpha f}{B \circ \alpha}(t), \quad t \in {}_{b-1}\mathbb{N}. \end{aligned} \quad (8)$$

Note that in Definitions 7 and 8, if $\alpha(t) \equiv 0$, then we recover the initial function; if $\alpha(t) \equiv 1$, then we recover the ordinary sum.

Definition 9 (*Left Riemann–Liouville AB nabla fractional differences of order $\alpha(t)$ — types I and II*) Let $0 < \alpha(t) < 1/2$ for all $t \in \mathbb{N}_a$. For a function $f : \mathbb{N}_a \rightarrow \mathbb{R}$,

1. the type *I* left Riemann–Liouville *AB* nabla fractional difference of order $\alpha(t)$ is defined by

$${}_a^{ABR}\nabla^{\alpha(t)} f(t) = \nabla \mathbf{E}_{\alpha(t), 1, \frac{-\alpha(t)}{1-\alpha(t)}, a^+} f(t), \quad t \in \mathbb{N}_{a+1}; \quad (9)$$

2. the type *II* left Riemann–Liouville *AB* nabla fractional difference of order $\alpha(t)$ is defined by

$${}_a^{ABR}\widehat{\nabla}^{\alpha(t)} f(t) = \nabla \mathcal{E}_{\alpha(t), 1, \frac{-\alpha(t)}{1-\alpha(t)}, a^+} f(t), \quad t \in \mathbb{N}_{a+1}. \quad (10)$$

Definition 10 (*Right Riemann–Liouville AB nabla fractional differences of order $\alpha(t)$ — types I and II*) Let $0 < \alpha(t) < 1/2$ for all $t \in {}_b\mathbb{N}$. For a function $f : {}_b\mathbb{N} \rightarrow \mathbb{R}$,

1. the type *I* right Riemann–Liouville *AB* nabla fractional difference of order $\alpha(t)$ is defined by

$${}^{ABR}\nabla_b^{\alpha(t)} f(t) = -\Delta \mathbf{E}_{\alpha(t), 1, \frac{-\alpha(t)}{1-\alpha(t)}, b^-} f(t), \quad t \in {}_{b-1}\mathbb{N};$$

2. the type *II* right Riemann–Liouville *AB* nabla fractional difference of order $\alpha(t)$ is defined by

$${}^{ABR}\widehat{\nabla}_b^{\alpha(t)} f(t) = -\Delta \mathcal{E}_{\alpha(t), 1, \frac{-\alpha(t)}{1-\alpha(t)}, b^-} f(t), \quad t \in {}_{b-1}\mathbb{N}.$$

Definition 11 (Left Caputo AB nabla fractional differences of order $\alpha(t)$ — types I and II) Let $0 < \alpha(t) < 1/2$ for all $t \in \mathbb{N}_a$. For a function $f : \mathbb{N}_a \rightarrow \mathbb{R}$,

1. the type I left Caputo AB nabla fractional difference of order $\alpha(t)$ is defined by

$${}_a^{ABC} \nabla^{\alpha(t)} f(t) = \mathbf{E}_{\alpha(t), 1, \frac{-\alpha(t)}{1-\alpha(t)}, a^+} \nabla f(t), \quad t \in \mathbb{N}_{a+1};$$

2. the type II left Caputo AB nabla fractional difference of order $\alpha(t)$ is defined by

$${}_a^{ABC} \widehat{\nabla}^{\alpha(t)} f(t) = \mathcal{E}_{\alpha(t), 1, \frac{-\alpha(t)}{1-\alpha(t)}, a^+} \nabla f(t), \quad t \in \mathbb{N}_{a+1}.$$

Definition 12 (Right Caputo AB nabla fractional differences of order $\alpha(t)$ — types I and II) Let $0 < \alpha(t) < 1/2$ for all $t \in {}_b\mathbb{N}$. For a function $f : {}_b\mathbb{N} \rightarrow \mathbb{R}$,

1. the type I right Caputo AB nabla fractional difference of order $\alpha(t)$ is defined by

$${}_b^{ABC} \nabla_b^{\alpha(t)} f(t) = -\mathbf{E}_{\alpha(t), 1, \frac{-\alpha(t)}{1-\alpha(t)}, b^-} \Delta f(t), \quad t \in {}_{b-1}\mathbb{N};$$

2. the type II right Caputo AB nabla fractional difference of order $\alpha(t)$ is defined by

$${}_b^{ABC} \widehat{\nabla}_b^{\alpha(t)} f(t) = -\mathcal{E}_{\alpha(t), 1, \frac{-\alpha(t)}{1-\alpha(t)}, b^-} \Delta f(t), \quad t \in {}_{b-1}\mathbb{N}.$$

Remark 1 If we replace $\alpha(t)$ in (1) and (3) by $\alpha(t-s)$ and replace each $\alpha(s)$ in (2) and (4) by $\alpha(t-s)$, then the ABR and ABC fractional differences with variable order can be expressed in convolution form. Similarly, if we replace $\alpha(t)$ in (5) and (7) by $\alpha(t-s)$ and replace each $\alpha(s)$ in (6) and (8) by $\alpha(t-s)$, then the second part of the AB fractional integrals with variable order can be expressed in convolution form.

3 Summation by Parts for Variable Order Fractional Operators

Summation/integration by parts has a very important role in mathematics: see, e.g., [31, 35, 45]. This is particularly true in the calculus of variations and optimal control, to prove necessary optimality conditions of Euler–Lagrange type (cf. proof of Theorem 3).

Lemma 1 (Integration by parts formula for nabla fractional sums of order $\alpha(t)$) Let $0 < \alpha(t) \leq 1$ for all $t \in \mathbb{N}_{a,b}$. For functions $f, g : \mathbb{N}_{a,b} \rightarrow \mathbb{R}$, we have

$$\sum_{t=a+1}^{b-1} f(t) {}_a \nabla^{-\alpha(t)} g(t) = \sum_{t=a+1}^{b-1} g(t) {}_b \nabla^{-\alpha(t)} f(t);$$

$$\sum_{t=a+1}^{b-1} f(t) \nabla_b^{-\alpha(t)} g(t) = \sum_{t=a+1}^{b-1} g(t) *_a \nabla^{-\alpha(t)} f(t).$$

Proof From Definition 3, and by changing the order of summation, we get

$$\begin{aligned} \sum_{t=a+1}^{b-1} f(t) {}_a \nabla^{-\alpha(t)} g(t) &= \sum_{t=a+1}^{b-1} f(t) \frac{1}{\Gamma(\alpha(t))} \sum_{s=a+1}^t (t - \rho(s))^{\overline{\alpha(t)-1}} g(s) \\ &= \sum_{s=a+1}^{b-1} g(s) \left(\sum_{t=s}^{b-1} (t - \rho(s))^{\overline{\alpha(t)-1}} f(t) \frac{1}{\Gamma(\alpha(t))} \right) \\ &= \sum_{s=a+1}^{b-1} g(s) *_b \nabla^{-\alpha(t)} f(s). \end{aligned}$$

The proof of the second assertion follows similarly.

Now, with the help of Lemma 1, we can prove the following integration by parts formula for AB fractional sums of variable order.

Theorem 1 (Integration by parts formula for AB nabla fractional sums of order $\alpha(t)$) *Let $0 < \alpha(t) \leq 1$ for all $t \in \mathbb{N}_{a,b}$. For functions $f, g : \mathbb{N}_{a,b} \rightarrow \mathbb{R}$, we have*

$$\begin{aligned} \sum_{t=a+1}^{b-1} f(t) {}_a^{AB} \nabla^{-\alpha(t)} g(t) &= \sum_{t=a+1}^{b-1} g(t) *_a^{AB} \nabla^{-\alpha(t)} f(t); \\ \sum_{t=a+1}^{b-1} f(t) *_a^{AB} \nabla^{-\alpha(t)} g(t) &= \sum_{t=a+1}^{b-1} g(t) {}^{AB} \nabla_b^{-\alpha(t)} f(t). \end{aligned}$$

Proof From Definition 7 and the first part of Lemma 1, we get

$$\begin{aligned} \sum_{t=a+1}^{b-1} f(t) {}_a^{AB} \nabla^{-\alpha(t)} g(t) &= \sum_{t=a+1}^{b-1} f(t) \frac{1 - \alpha(t)}{B(\alpha(t))} g(t) + \sum_{t=a+1}^{b-1} f(t) \frac{\alpha(t)}{B(\alpha(t))} {}_a \nabla^{-\alpha(t)} g(t) \\ &= \sum_{t=a+1}^{b-1} f(t) \frac{1 - \alpha(t)}{B(\alpha(t))} g(t) + \sum_{t=a+1}^{b-1} g(t) *_b \nabla^{-\alpha(t)} \frac{\alpha f}{B \circ \alpha}(t) \\ &= \sum_{t=a+1}^{b-1} g(t) \left(\frac{1 - \alpha(t)}{B(\alpha(t))} f(t) + *_b \nabla^{-\alpha(t)} \frac{\alpha f}{B \circ \alpha}(t) \right) \\ &= \sum_{t=a+1}^{b-1} g(t) *_a^{AB} \nabla^{-\alpha(t)} f(t). \end{aligned}$$

The proof of the second assertion is similar to the first one. It follows from Definition 7 and the second part of Lemma 1.

Lemma 2 Let $0 < \alpha(t) < 1/2$ for all $t \in \mathbb{N}_{a,b}$. For functions $f, g : \mathbb{N}_{a,b} \rightarrow \mathbb{R}$, we have

$$\begin{aligned} \sum_{t=a+1}^{b-1} f(t) E_{\alpha(t), 1, \frac{-\alpha(t)}{1-\alpha(t)}, a^+} g(t) &= \sum_{t=a+1}^{b-1} g(t) \mathcal{E}_{\alpha(t), 1, \frac{-\alpha(t)}{1-\alpha(t)}, b^-} f(t); \\ \sum_{t=a+1}^{b-1} f(t) \mathcal{E}_{\alpha(t), 1, \frac{-\alpha(t)}{1-\alpha(t)}, a^+} g(t) &= \sum_{t=a+1}^{b-1} g(t) E_{\alpha(t), 1, \frac{-\alpha(t)}{1-\alpha(t)}, b^-} f(t). \end{aligned}$$

Proof From Definitions 5 and 6, and by changing the order of summation, we have

$$\begin{aligned} \sum_{t=a+1}^{b-1} f(t) E_{\alpha(t), 1, \frac{-\alpha(t)}{1-\alpha(t)}, a^+} g(t) &= \sum_{t=a+1}^{b-1} f(t) \frac{B(\alpha(t))}{1-\alpha(t)} \sum_{s=a+1}^t E_{\alpha(t)} \left[\frac{-\alpha(t)}{1-\alpha(t)}, t - \rho(s) \right] g(s) \\ &= \sum_{s=a+1}^{b-1} g(s) \sum_{t=s}^{b-1} \frac{B(\alpha(t))}{1-\alpha(t)} E_{\alpha(t)} \left[\frac{-\alpha(t)}{1-\alpha(t)}, t - \rho(s) \right] f(t) \\ &= \sum_{s=a+1}^{b-1} g(s) \mathcal{E}_{\alpha(s), 1, \frac{-\alpha(s)}{1-\alpha(s)}, b^-} f(s). \end{aligned}$$

The proof of the second assertion follows similarly.

Theorem 2 Let $0 < \alpha(t) < 1/2$ for all $t \in \mathbb{N}_{a,b}$. For functions $f, g : \mathbb{N}_{a,b} \rightarrow \mathbb{R}$, we have

$$\begin{aligned} \sum_{t=a+1}^{b-1} f(t) {}^a ABC \nabla^{\alpha(t)} g(t) &= g(t) \mathcal{E}_{\alpha(t), 1, \frac{-\alpha(t)}{1-\alpha(t)}, b^-} f(t) \Big|_a^{b-1} + \sum_{t=a+1}^{b-1} g(t-1) {}^a BR \widehat{\nabla}_b^{\alpha(t)} f(t-1); \\ \sum_{t=a+1}^{b-1} f(t) {}^a ABC \widehat{\nabla}^{\alpha(t)} g(t) &= g(t) E_{\alpha(t), 1, \frac{-\alpha(t)}{1-\alpha(t)}, b^-} f(t) \Big|_a^{b-1} + \sum_{t=a+1}^{b-1} g(t-1) {}^a BR \nabla_b^{\alpha(t)} f(t-1); \\ \sum_{t=a+1}^{b-1} f(t) {}^a ABC \nabla_b^{\alpha(t)} g(t) &= -g(t) \mathcal{E}_{\alpha(t), 1, \frac{-\alpha(t)}{1-\alpha(t)}, a^+} f(t) \Big|_{a+1}^b + \sum_{t=a+1}^{b-1} g(t+1) {}^a BR \widehat{\nabla}^{\alpha(t)} f(t+1); \\ \sum_{t=a+1}^{b-1} f(t) {}^a ABC \widehat{\nabla}_b^{\alpha(t)} g(t) &= -g(t) E_{\alpha(t), 1, \frac{-\alpha(t)}{1-\alpha(t)}, a^+} f(t) \Big|_{a+1}^b + \sum_{t=a+1}^{b-1} g(t+1) {}^a BR \nabla^{\alpha(t)} f(t+1). \end{aligned}$$

Proof We will only prove the first assertion. The proof of the others follow similarly. From Definitions 10 and 11, the first part of Lemma 2 and the summation by parts formula from ordinary difference calculus, we get

$$\begin{aligned}
& \sum_{t=a+1}^{b-1} f(t) {}^ABC \nabla^{\alpha(t)} g(t) \\
&= \sum_{t=a+1}^{b-1} f(t) \mathbf{E}_{\alpha(t), 1, \frac{-\alpha(t)}{1-\alpha(t)}, a^+} \nabla g(t) \\
&= \sum_{t=a+1}^{b-1} \nabla g(t) \mathcal{E}_{\alpha(t), 1, \frac{-\alpha(t)}{1-\alpha(t)}, b^-} f(t) \\
&= g(t) \mathcal{E}_{\alpha(t), 1, \frac{-\alpha(t)}{1-\alpha(t)}, b^-} f(t)|_a^{b-1} - \sum_{t=a+1}^{b-1} g(t-1) \nabla \mathcal{E}_{\alpha(t), 1, \frac{-\alpha(t)}{1-\alpha(t)}, b^-} f(t) \\
&= g(t) \mathcal{E}_{\alpha(t), 1, \frac{-\alpha(t)}{1-\alpha(t)}, b^-} f(t)|_a^{b-1} - \sum_{t=a+1}^{b-1} g(t-1) \Delta \mathcal{E}_{\alpha(t), 1, \frac{-\alpha(t)}{1-\alpha(t)}, b^-} f(t-1) \\
&= g(t) \mathcal{E}_{\alpha(t), 1, \frac{-\alpha(t)}{1-\alpha(t)}, b^-} f(t)|_a^{b-1} + \sum_{t=a+1}^{b-1} g(t-1) {}^{ABR} \widehat{\nabla}_b^{\alpha(t)} f(t-1).
\end{aligned}$$

The proof is complete.

4 Variable Order Fractional Variational Principles

The fractional calculus of variations of variable-order is a subject under strong current development [5, 49]. However, to the best of our knowledge, available results are only for the continuous time scale $\mathbb{T} = \mathbb{R}$. Here we obtain the main result of a variational calculus, that is, an Euler–Lagrange necessary optimality condition, for the isolated time scale $\mathbb{T} = \mathbb{N}_{a+1, b-1}$.

Let J be a functional of the form

$$J(f) = \sum_{t=a+1}^{b-1} L(t, f^\rho(t), {}^ABC \nabla^{\alpha(t)} f(t)),$$

where $0 < \alpha(t) < 1/2$ for all $t \in \mathbb{N}_{a+1, b-1}$, $f : \mathbb{N}_{a, b-1} \rightarrow \mathbb{R}$ and $L : \mathbb{N}_{a+1, b-1} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$.

Theorem 3 *Let f be a local extremizer of J satisfying the boundary conditions*

$$f(a) = A, \quad f(b-1) = B.$$

Then f satisfies the Euler–Lagrange equation

$$L_1^\sigma(t) + {}^{ABR} \widehat{\nabla}_b^{\alpha(t)} L_2(t) = 0, \quad t \in \mathbb{N}_{a+1, b-2},$$

where $L_1 = \frac{\partial L}{\partial f^\rho}$ and $L_2 = \frac{\partial L}{\partial {}^{ABC}\nabla^{\alpha(t)} f}$.

Proof Let ε be a small real parameter and $\eta : \mathbb{N}_{a,b-1} \rightarrow \mathbb{R}$ be a function such that $\eta(a) = \eta(b-1) = 0$. Consider a variation of f , say $f + \varepsilon\eta$. Since the Caputo difference operator ${}^{ABC}\nabla^{\alpha(t)}$ is linear, it follows that

$$J(f + \varepsilon\eta) = \sum_{t=a+1}^{b-1} L(t, f^\rho(t) + \varepsilon\eta^\rho(t), {}^{ABC}\nabla^{\alpha(t)} f(t) + \varepsilon {}^{ABC}\nabla^{\alpha(t)} \eta(t)).$$

Define $\hat{J}(\varepsilon) = J(f + \varepsilon\eta)$. Because f is a local extremizer of J , $\hat{J}$ attains a local extremum at $\varepsilon = 0$. Differentiating $\hat{J}(\varepsilon)$ at zero, we get

$$\begin{aligned} & \sum_{t=a+1}^{b-1} \eta^\rho(t) \frac{\partial L}{\partial f^\rho}(t, f^\rho(t), {}^{ABC}\nabla^{\alpha(t)} f(t)) \\ & + {}^{ABC}\nabla^{\alpha(t)} \eta(t) \frac{\partial L}{\partial {}^{ABC}\nabla^{\alpha(t)} f}(t, f^\rho(t), {}^{ABC}\nabla^{\alpha(t)} f(t)) = 0. \end{aligned}$$

Using the first integration by parts formula in Theorem 2, we have

$$\begin{aligned} & \sum_{t=a+1}^{b-1} \eta^\rho(t) \left[\frac{\partial L}{\partial f^\rho}(t, f^\rho(t), {}^{ABC}\nabla^{\alpha(t)} f(t)) \right. \\ & \left. + \left({}^{ABR}\widehat{\nabla}_b^{\alpha(t)} \frac{\partial L}{\partial {}^{ABC}\nabla^{\alpha(t)} f}(t, f^\rho(t), {}^{ABC}\nabla^{\alpha(t)} f(t)) \right) (t-1) \right] \\ & + \eta(t) \left(\mathcal{E}_{\alpha(t), 1, \frac{-\alpha(t)}{1-\alpha(t)}, b^-} \frac{\partial L}{\partial {}^{ABC}\nabla^{\alpha(t)} f}(t, f^\rho(t), {}^{ABC}\nabla^{\alpha(t)} f(t)) \right) (t) \Big|_a^{b-1} = 0. \end{aligned}$$

Since $\eta(a) = \eta(b-1) = 0$ and η is arbitrary, it follows that

$$\frac{\partial L}{\partial f^\rho}(t, f^\rho(t), {}^{ABC}\nabla^{\alpha(t)} f(t)) + \left({}^{ABR}\widehat{\nabla}_b^{\alpha(t)} \frac{\partial L}{\partial {}^{ABC}\nabla^{\alpha(t)} f}(t, f^\rho(t), {}^{ABC}\nabla^{\alpha(t)} f(t)) \right) (t-1) = 0$$

for all $t \in \mathbb{N}_{a+2, b-1}$.

Although we only consider here a class of fractional variable order variational problems (FVOVP), our Theorem 3 can be easily extended to many other related FVOVPs involving the new variable-order fractional differences introduced in Sect. 2. We trust that this observation will initiate some interest in further future developments.

Acknowledgements Abdeljawad is grateful to Prince Sultan University for funding this work through research group *Nonlinear Analysis Methods in Applied Mathematics* (NAMAM), number RG-DES-2017-01-17; Torres to the support of FCT within the R&D unit CIDMA, UID/MAT/04106/2019.

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